MTH 166

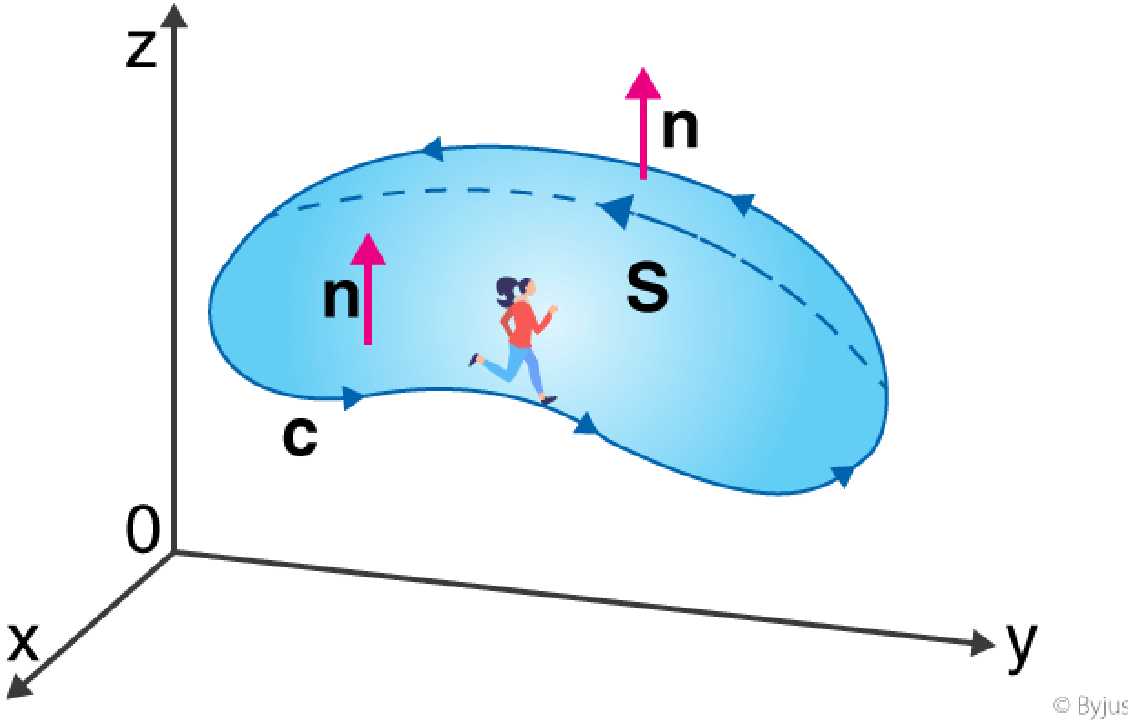
Lecture-33

Stokes' Theorem

Statement:

Let S be a piecewise smooth orientable surface bounded by a piecewise smooth simple closed curve c traced in a positive direction. Let V be a vector which is continuous and have continuous first order partial derivatives. Let n is a unit normal vector drawn outward to the surface S. Then:

$$\oint_C \vec{V}.\overrightarrow{dr} = \int_S \int (Curl\,\vec{V}).\hat{n}dA = \int_S \int (\vec{\nabla} \times \vec{V}).\hat{n}dA$$



Calculations required to Apply Stokes' Theorem

1. Calculate $Curl\ \vec{V} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ V_1 & V_2 & V_3 \end{vmatrix}$

2. Calculate $\hat{n} = \frac{\vec{\nabla} f}{|\vec{\nabla} f|} = \frac{grad\ f}{|grad\ f|}$

3. Calculate dA : By taking projection of given surface S

If projection is taken on xy-plane (z=0 plane), then: $dA = \frac{dxdy}{\hat{n}.\hat{k}}$

If projection is taken on yz-plane (x=0 plane), then: $dA = \frac{dydz}{\hat{n}.\hat{\imath}}$

If projection is taken on zx-plane (y=0 plane), then: $dA = \frac{dzdx}{\hat{n}.\hat{\jmath}}$

****** Stokes' theorem gives a relationship between line integral and double integral like that of Green's Theorem

Problem 1: Evaluate $\oint_C \vec{V}.\overrightarrow{dr}$ using Stokes' theorem where $\vec{V} = x\hat{\imath} + y\hat{\jmath} + z\hat{k}$ and S is the surface bounded by sphere $x^2 + y^2 + z^2 = 16$.

Solution: Here $\vec{V} = x\hat{\imath} + y\hat{\jmath} + z\hat{k}$

1. $Curl\,\vec{V} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ V_1 & V_2 & V_3 \end{vmatrix} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x & y & z \end{vmatrix} = \vec{0}$

By Stokes' theorem:

$$\oint_C \vec{V}.\overrightarrow{dr} = \int_S \int \left(Curl\,\vec{V}\right).\hat{n}dA = 0$$

* * This is how an MCQ can be framed on Stokes' theorem.

Problem 2: Show that $\oint_C \vec{V}.\overrightarrow{dr}$ is always zero ,using Stokes' theorem where $\vec{V} = yz\hat{\imath} + zx\hat{\jmath} + xy\hat{k}$ and S is bounded by any closed curve c.

Solution: Here $\vec{V} = yz\hat{\imath} + zx\hat{\jmath} + xy\hat{k}$

$$\textbf{1. } Curl\,\vec{V} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ V_1 & V_2 & V_3 \end{vmatrix} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ yz & zx & xy \end{vmatrix}$$

$$= (x - x)\hat{\imath} - (y - y)\hat{\jmath} + (z - z)\hat{k} = \vec{0}$$

By Stokes' theorem:

$$\oint_C \vec{V}.\overrightarrow{dr} = \int_S \int \left(Curl\,\vec{V}\right).\hat{n}\,dA = 0$$

* * This is how an MCQ can be framed on Stokes' theorem.

Problem 3: Evaluate $\oint_C \vec{V}.\overrightarrow{dr}$ by Stokes' theorem where $\vec{V} = (3x - y)\hat{\imath} - 2yz^2\hat{\jmath} - 2y^2z\hat{k}$ and S is the surface bounded by sphere $x^2 + y^2 + z^2 = 16, z > 0$.

Solution: Here $\vec{V} = (3x - y)\hat{\imath} - 2yz^2\hat{\jmath} - 2y^2z\hat{k}$

1. $Curl\ \vec{V} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ V_1 & V_2 & V_3 \end{vmatrix} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ (3x - y) & -2yz^2 & -2y^2z \end{vmatrix} = 0\hat{\imath} + 0\hat{\jmath} + 1\hat{k}$

$Curl\ \vec{V} = 0\hat{\imath} + 0\hat{\jmath} + 1\hat{k}$

2. $\hat{n} = \frac{\vec{\nabla} f}{|\vec{\nabla} f|} = \frac{grad\ f}{|grad\ f|}$

Let $f = x^2 + y^2 + z^2 = 16\ where\ z > 0$

$\Longrightarrow f_x = 2x, f_y = 2y, f_z = 2z$

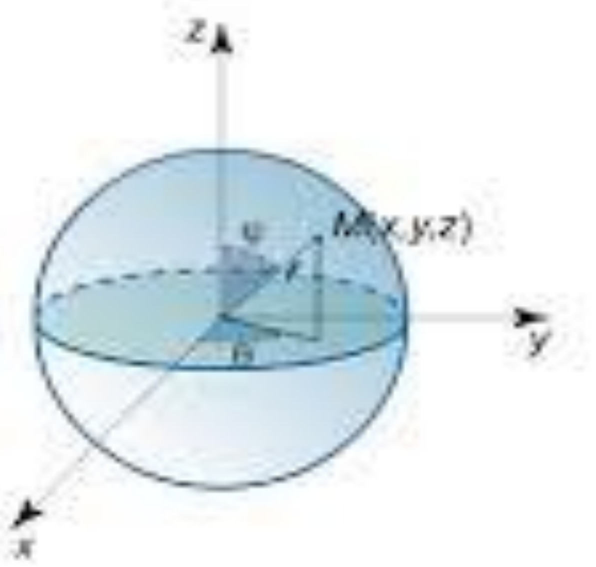
Now, $\vec{\nabla} f = f_x\hat{\imath} + f_y\hat{\jmath} + f_z\hat{k} = 2x\hat{\imath} + 2y\hat{\jmath} + 2z\hat{k}$

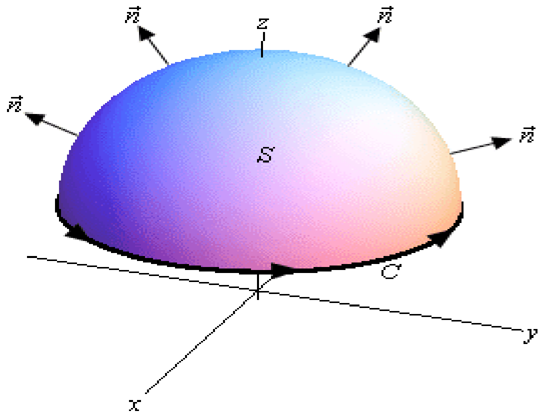
$$\hat{n} = \frac{\vec{\nabla} f}{|\vec{\nabla} f|} = \frac{(2x\hat{\imath}+2y\hat{\jmath}+2z\hat{k})}{\sqrt{4x^2+4y^2+4z^2}}$$

$$\hat{n} = \frac{2(x\hat{\imath}+y\hat{\jmath}+z\hat{k})}{2\sqrt{x^2+y^2+z^2}} = \frac{(x\hat{\imath}+y\hat{\jmath}+z\hat{k})}{\sqrt{16}}$$

$$\hat{n} = \frac{(x\hat{\imath}+y\hat{\jmath}+z\hat{k})}{4}$$

3. To find dA: Let the projection of surface S be taken on xy-plane.(z=0) The sphere becomes a circle of radius 4, $x^2 + y^2 + 0^2 = 16$ that is $x^2 + y^2 = 16$





$$dA = \frac{dxdy}{\hat{n}.\hat{k}} = \frac{dxdy}{\frac{(x\hat{\imath}+y\hat{\jmath}+z\hat{k})}{4}.\hat{k}}$$

$$dA = \frac{dxdy}{\frac{(z)}{4}}$$

By Stokes' Theorem:

$$\oint_C \vec{V}.\overrightarrow{dr} = \int_S \int \left(Curl\,\vec{V}\right).\hat{n}dA$$

$$= \int_S \int \left[\left(0\hat{\imath} + 0\hat{\jmath} + 1\hat{k}\right).\frac{(x\hat{\imath}+y\hat{\jmath}+z\hat{k})}{4}\right]\frac{dxdy}{\frac{(z)}{4}}$$

$$= \int_S \int \frac{(z)}{4}\frac{dxdy}{\frac{(z)}{4}} = \int_S \int dxdy = \text{Area of a circle } x^2 + y^2 = 16$$

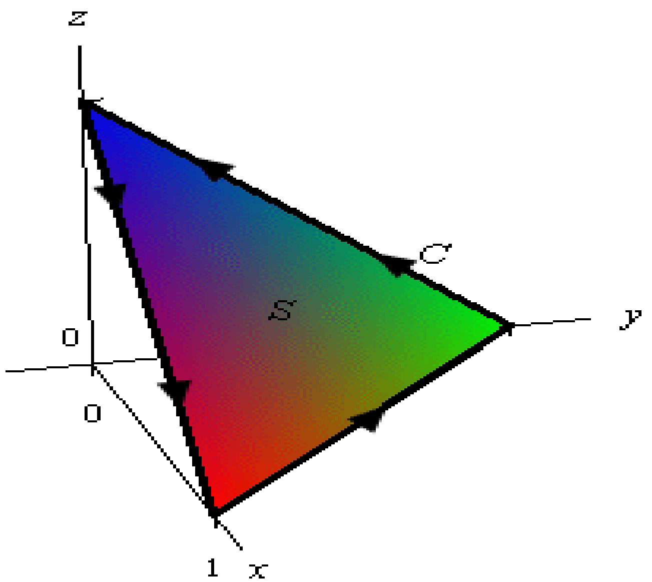
$$= \pi r^2 = \pi(4^2) = 16\pi$$ **Answer.**

Problem 4: Evaluate $\oint_C \vec{V}.\overrightarrow{dr}$ by Stokes' theorem where $\vec{V} = z^2\hat{\imath} + y^2\hat{\jmath} + x\,\hat{k}$ and S is the surface bounded by triangle with the vertices $(1,0,0), (0,1,0), (0,0,1)$.

Solution: Here $\vec{V} = z^2\hat{\imath} + y^2\hat{\jmath} + x\,\hat{k}$

1. $Curl\,\vec{V} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ V_1 & V_2 & V_3 \end{vmatrix} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ z^2 & y^2 & x \end{vmatrix} = 0\hat{\imath} + (2z-1)\hat{\jmath} + 1\hat{k}$

$Curl\,\vec{V} = 0\hat{\imath} + (2z-1)\hat{\jmath} + 1\hat{k}$



2. $\hat{n} = \frac{\vec{\nabla} f}{|\vec{\nabla} f|} = \frac{grad\ f}{|grad\ f|}$

Let $f = x + y + z = 1$

$\Longrightarrow f_x = 1, f_y = 1, f_z = 1$

Now, $\vec{\nabla} f = f_x\hat{\imath} + f_y\hat{\jmath} + f_z\hat{k} = \hat{\imath} + \hat{\jmath} + \hat{k}$

$$\hat{n} = \frac{\vec{\nabla} f}{|\vec{\nabla} f|} = \frac{(\hat{\imath}+\hat{\jmath}+\hat{k})}{\sqrt{1+1+1}}$$

$$\hat{n} = \frac{(\hat{\imath}+\hat{\jmath}+\hat{k})}{\sqrt{3}}$$

3. To find dA: Let the projection of surface S be taken on xy-plane.(z=0)

$$z = 1 - x - y$$

$$dA = \frac{dxdy}{\hat{n}.\hat{k}} = \frac{dxdy}{\frac{(\hat{\imath}+\hat{\jmath}+\hat{k})}{\sqrt{3}}.\hat{k}}$$

$$dA = \frac{dxdy}{\frac{1}{\sqrt{3}}}$$

By Stokes' Theorem:

$$\oint_C \vec{V}.\overrightarrow{dr} = \int_S \int (Curl\ \vec{V}).\hat{n}dA$$

$$= \int_S \int [(0\hat{\imath} + (2z-1)\hat{\jmath} + 1\hat{k}).\frac{(\hat{\imath}+\hat{\jmath}+\hat{k})}{\sqrt{3}}]\frac{dxdy}{\frac{(1)}{\sqrt{3}}}$$

$$= \int_S \int [2(1-x-y)-1]\,dxdy$$

$$= \int_{x=0}^{1} \int_{y=0}^{1-x} [2(1-x-y)-1]dydx$$

$$= \int_{x=0}^{1} [y-2xy-y^2]_{y=0}^{y=1-x} dx$$

$$= \int_{x=0}^{1} (x^2-x)dx$$

$$= \left[\frac{x^3}{3}-\frac{x^2}{2}\right]_{x=0}^{1}$$

$$= \frac{1}{3}-\frac{1}{2}$$

$= \frac{1}{6}$ **<u>Answer</u>**

Problem 5: Evaluate $\oint_{C_2} \vec{V}.\overrightarrow{dr}$ by Stokes' theorem where $\vec{V} = z\hat{\imath} + x\hat{\jmath} + z\hat{k}$ and S is the portion of the sphere $x^2 + y^2 + z^2 = 9$ above xy-plane.

Solution: Here $\vec{V} = z\hat{\imath} + x\hat{\jmath} + z\hat{k}$

1. $Curl\ \vec{V} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ V_1 & V_2 & V_3 \end{vmatrix} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ z & x & z \end{vmatrix} = 0\hat{\imath} + 1\hat{\jmath} + 1\hat{k}$

$Curl\ \vec{V} = 0\hat{\imath} + 1\hat{\jmath} + 1\hat{k}$

2. $\hat{n} = \frac{\vec{\nabla} f}{|\vec{\nabla} f|} = \frac{grad\ f}{|grad\ f|}$

Let $f = x^2 + y^2 + z^2 = 9$ above xy-plane.

$\Longrightarrow f_x = 2x, f_y = 2y, f_z = 2z$

Now, $\vec{\nabla} f = f_x\hat{\imath} + f_y\hat{\jmath} + f_z\hat{k} = 2x\hat{\imath} + 2y\hat{\jmath} + 2z\hat{k}$

$$\hat{n} = \frac{\vec{\nabla} f}{|\vec{\nabla} f|} = \frac{(2x\hat{\imath}+2y\hat{\jmath}+2z\hat{k})}{\sqrt{4x^2+4y^2+4z^2}}$$

$$\hat{n} = \frac{2(x\hat{\imath}+y\hat{\jmath}+z\hat{k})}{2\sqrt{x^2+y^2+z^2}} = \frac{(x\hat{\imath}+y\hat{\jmath}+z\hat{k})}{\sqrt{9}}$$

$$\hat{n} = \frac{(x\hat{\imath}+y\hat{\jmath}+z\hat{k})}{3}$$

3. To find dA: Let the projection of surface S be taken on xy-plane.(z=0) The sphere becomes a circle of radius 4, $x^2 + y^2 + 0^2 = 9$ that is $x^2 + y^2 = 3$

$$dA = \frac{dxdy}{\hat{n}.\hat{k}} = \frac{dxdy}{\frac{(x\hat{\imath}+y\hat{\jmath}+z\hat{k})}{3}.\hat{k}}$$

$$dA = \frac{dxdy}{\frac{(z)}{3}}$$

By Stokes' Theorem:

$$\oint_C \vec{V}.\overrightarrow{dr} = \int_S \int \left(Curl\,\vec{V}\right).\hat{n}dA$$

$$= \int_S \int [\left(0\hat{\imath} + 1\hat{\jmath} + 1\hat{k}\right).\frac{\left(x\hat{\imath}+y\hat{\jmath}+z\hat{k}\right)}{3}] \frac{dxdy}{\frac{(z)}{3}}$$

$$= \int_S \int \frac{(y+z)}{3} \frac{dxdy}{\frac{(z)}{3}} = \int_S \int \frac{(y+z)}{z} dxdy$$

$$= \int_S \int \left(\frac{y}{z} + 1\right) dxdy$$

$$= \int_S \int \frac{y}{z} dxdy + \int_S \int dxdy$$

$$== \int_S \int \frac{y}{\sqrt{9-(x^2+y^2)}} dxdy + \text{Area of a circle with radius 3}$$

$=$(Change into polar coordinates) + $\pi(3^2)$

$$= (x = r\cos\theta, y = r\sin\theta, 0 \le r \le 3, 0 \le \theta \le 2\pi) + 9\pi$$

$$= \int_{r=0}^{3}\int_{\theta=0}^{2\pi} \frac{r\sin\theta}{\sqrt{9-r^2}} r dr d\theta + 9\pi$$

$$= \int_{r=0}^{3} \frac{r^2}{\sqrt{9-r^2}} dr \int_{\theta=0}^{2\pi} \sin\theta \, d\theta + 9\pi$$

Simplify it.